April 1, 2026

Problem Set 3

Exercise 1: $\mathbf{P} = \mathbf{NP}$, or does it?. Prove the following:

1. ($\square$) If $\mathbf{P} = \mathbf{PSPACE}$, then $\mathbf{P} = \mathbf{NP}$.
2. ($\square$) If $\mathbf{NTIME}(f(n)) \subseteq \mathbf{DTIME}(f^6(n)(\log(f(n)))^7)$, then $\mathbf{P} = \mathbf{NP}$.
3. ($\star$) If $\Sigma_{67} = \Pi_0$, then $\mathbf{P} = \mathbf{NP}$.
4. ($\star$) If $\mathbf{P} = \mathbf{P/poly}$, then $\mathbf{P} = \mathbf{NP}$.
5. ($\star\star$) If $\mathbf{P} = \mathbf{P/log}$, then $\mathbf{P} = \mathbf{NP}$.
6. ($\star\star$) If $\mathbf{P} = \mathbf{LOGSPACE}$, then $\mathbf{P} = \mathbf{NP}$.
7. ($\star\star\star$) If $\mathbf{NP} \subseteq \mathbf{BPP}$ and $\mathbf{RP} = \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.
8. ($\star\star\star$) If $\mathbf{PCP}[0, O(\log)] = \mathbf{PCP}[O(\log n), O(1)]$, then $\mathbf{P} = \mathbf{NP}$.
9. ($\star\star\star\star$) If there is no $\mathbf{NP}$-intermediate language, then $\mathbf{P} = \mathbf{NP}$.
10. ($\star\star\star\star$) If there is no $\mathbf{BPP}$-complete language and $\mathbf{NP} = \mathbf{RP}$, then $\mathbf{P} = \mathbf{NP}$.

Exercise 2: An assortment of problems.

1. ($\star\star\star$) Prove by induction that every real number is finite.

2. ($\star$) Suppose we have two sets of tiles: $A = \{a, b, c, d, e, f\}$ and $B = \{1, 2, \ldots, 10, 11\}$. We are allowed to choose one tile from each set in order to communicate a message to a friend (we can communicate with this friend prior to establish a communication scheme). What is the smallest number of distinct messages where we cannot devise a protocol to fully pairwise distinguish them using these tiles.

3. ($\star\star$) Describe the set of languages recognized by DFAs with an *infinite* number of states.

4. ($\star$) Prove that if a *directed acyclic graph* has 67 connected components, it does in fact contain a cycle.

5. ($\star\star$) Suppose that we have discrete probability measures μ and ν on $\mathbb{R}^d$. Consider $\theta_1, \theta_2, \ldots, \theta_d \in S^{d-1}$. Let $\theta^*\mu$ and $\theta^*\nu$ be the linear projections onto θ. Show that if $\theta_i^*\mu = \theta_i^*\nu$ for all $i \in [d]$, then $\mu = \nu$.

6. ($\star$) Argue that multiplying $n \times n$ matrices is $\Omega(n^2)$.

7. ($\star\star$) Define a *palindromic partition* of n to be a set such that $\sum\{a_1, a_2, \ldots, a_k\} = n$ and there exists permutation σ such that $a_{\sigma(1)}a_{\sigma(2)} \ldots a_{\sigma(k)}$ is a palindrome. Count the number of such partitions of the number 16. Do the same for 17. Hint: for 16, writing a short program may be helpful.

8. ($\star\star\star\star$) Prove that there does not exist $k > 1$ such that 2^k is prime. Examine whether this result holds for 3^k.

9. ($\star\star$) Take n points in $\mathbb{R}^{67}$. Prove that, in general, we cannot embed them into $\mathbb{R}$ such that the pairwise distance are preserved up to a $\log(67)$ multiplicative factor. Suitably generalize this to embedding arbitrary metric spaces into $\mathbb{R}$. Similarly argue that we cannot do this for metrics induced by a graph.

10. ($\star\star$) Let G be a finite non-abelian group. Consider the following game:

    - Take arbitrary $g \in G$ to be our starting state.
    - Player A and B both get random $H \trianglelefteq G$ as their set of valid moves. The players take turns multiplying the current state by $h \in H$ of their choice to obtain the next state (we multiply along the right).
    - A player wins if the current state is the identity on their turn.

    Argue that under optimal play, it is impossible for B to lose.

    Bonus: Consider a variant of chess where players must move two times in a row. Show that White cannot lose.